Spectral properties of the gauge invariant
quark Green's function in two-dimensional QCD

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Abstract

The gauge invariant quark Green's function with a path-ordered phase factor along a straight-line is studied in two-dimensional QCD in the large-N_c limit by means of an exact integrodifferential equation. Its spectral functions are analytically determined. They are infra-red finite and lie on the positive real axis of the complex plane of the momentum squared variable, corresponding to momenta in the forward light cone. Their singularities are represented by an infinite number of threshold type branch points with power-law $-3/2$, starting at positive mass values, characterized by an integer number n and increasing with n. The analytic expression of the Green's function for all momenta is presented. The appearance of strong threshold singularities is suggestive of the fact that quarks could not be observed as asymptotic states.

1 Introduction

Conventional quark Green's functions suffer from lack of gauge invariance. In particular, no physical conclusion can be drawn from any of their individual properties. Their role is rather limited to lead, together with other Green's functions, to the evaluation of gauge invariant observable quantities, like bound state masses, current matrix elements, hadronic form factors, scattering amplitudes, etc. This is not the case of the gauge invariant quark Green's functions (GIQGF), constructed with the aid of the gluon field path-ordered phase factors [1, 2]. Their properties should provide a more stable basis for the investigation of the physical characteristics of the underlying quark and gluon fields. Nevertheless, because of the mathematical complexity of the corresponding composite objects, not much has been known up to now about them.

Recently, the present author could obtain, using a representation for the quark propagator in an external gluon field that generalizes the one introduced in the nonrelativistic case [3], exact integrodifferential equations for the the two-point GIQGFs (2PGIQGF), in which the paths of the phase factors are of the skew-polygonal type [4]. It turns out that such GIQGFs form a closed set with respect to the above equations. The kernels of these equations are represented by functional derivatives of Wilson loop averages [5, 6, 7, 8, 9]. Designating by $S_{(1)}(x, x') \equiv S(x, x')$ the 2PGIQGF with a path made of one straight line joining the quark to the antiquark and by $S_{(n)}(x, x'; y_{n-1}, \ldots, y_1)$ ($n \geq 2$) the 2PGIQGF with a path made of n segments and $n - 1$ junction points $y_1, y_2, \ldots, y_{n-1}$ between the segments, the equation satisfied by S has the following structure:

$$(i\gamma.\partial_{(x)} - m)S(x, x') = i\delta^4(x - x') + i\gamma^\mu \left\{ K_{2\mu}(x', x, y_1) \, S_{(2)}(y_1, x'; x) \right.$$

$$\left. + \sum_{i=3}^{\infty} K_{i\mu}(x', x, y_1, \ldots, y_{i-1}) \, S_{(i)}(y_{i-1}, x'; x, y_1, \ldots, y_{i-2}) \right\}. \tag{1.1}$$

[m is the quark mass parameter. Integrations on the intermediate variables y are implicit.] The kernels K_n ($n = 2, \ldots, \infty$) contain logarithms of Wilson loop averages along the contour of the $(n + 1)$-sided skew-polygon with at most n functional derivatives along the contour and $(n - 1)$ GIQGFs S along the segments of the contour. The 2PGIQGFs $S_{(n)}$ ($n \geq 2$) are themselves related to the lowest-order 2PGIQGF S through functional relations involving mainly the Wilson loop average along the $(n + 1)$-sided skew-polygon. It is then conceivable that, by eliminating the $S_{(n)}$s in favor of S, one might reach a closed form equation for S.

What kind of information could provide us the analysis of the 2PGIQGF S? The status of that Green's function is rather intricate. If colored objects are confined and if singularities of Green's functions result from insertions of complete sets of physical states, which are here composed of hadronic states (color singlets), then it is not possible

to cut the line joining the quark to the antiquark and saturate it with physical states, the contribution of the latter being identically zero. This situation would suggest that the above Green's function is free of singularities. Nevertheless, the equation that it satisfies and which is derived from the QCD lagrangian signals the presence of singularities, generated by the free quark propagator. This paradoxical situation is solved only with the acceptance that the building blocks of the theory, which are the quark and gluon fields, continue forming a complete set of states with positive energies and could be used for any saturation scheme of intermediate states. It is then up to the theory to indicate us through the solutions of the equations of motion how these singularities combine to form the complete contributions. For sectors of colored states, we should expect to find singularities, whenever they are present, which would be stronger than the observable ones. Therefore, the determination of the 2PGIQGF would provide us a direct test of the confinement mechanism in the colored sector of quarks.

Saturating the intermediate states of the 2PGIQGF S with quarks and gluons with positive energies, one arrives at a generalized form of the Källen-Lehmann representation [4]:

$$S(p) = i \int_0^\infty ds \sum_{n=1}^\infty \frac{[\gamma.p\,\rho_1^{(n)}(s) + \rho_0^{(n)}(s)]}{(p^2 - s + i\varepsilon)^n}. \tag{1.2}$$

[$S(p)$ is the Fourier transform of $S(x, x') = S(x - x')$.] The increasing powers of the denominators come from the increasing number of gluon fields in the series expansion of the phase factor in terms of the coupling constant. Formally, one might reduce, by integrations by parts, all the denominators to a single one, but the possible presence of strong threshold singularities might stop the operation at some power. In the latter case, another alternative would be the extraction out of the integral of derivation operators.

There might also exist other extreme cases. For instance, the singularities of S, as provided by the solution of Eq. (1.1), might lie in unphysical regions, like the complex plane or the negative axis of p^2. Or, the summation of the series (1.2) might lead to new singularities in unphysical regions. We shall, however, continue exploring the optimistic scenario, according to which the series of singularities (1.2) are the only type of singularities of S such that their sum remains on the physical timelike axis of p^2.

This paper is devoted to the resolution of Eq. (1.1) in two-dimensional QCD in the large-N_c limit [10, 11, 12] . That theory displays the main features of confinement of the four-dimensional case with the additional simplification that asymptotic freedom is realized here in a rather trivial way, since the theory is superrenormalizable. Furthermore, many aspects of two-dimensional theories are often exactly solvable and in this case one might hope to have an unambiguous check of the physical implications of Eq. (1.1) for the spectral properties of the 2PGIQGF.

The main results of the present investigation can be summarized as follows. The series

of spectral functions of the vector and scalar types of Eq. (1.2) can actually be grouped into two single spectral functions. The singularities of the latter can be analytically determined. They are represented, by an infinite number of threshold type branch points with power-law $-3/2$, starting at positive mass values characterized by an integer number n and increasing with n. This situation is qualitatively unchanged in the limit of massless quarks. The strong thereshold singularities, together with the absence of pole terms, are suggestive of the fact that quarks could not be observed as physical asymptotic states.

The plan of the paper is the following. In Sec. 2, we consider two-dimensional QCD in the large-N_c limit. In Sec. 3, the Green's function is considered in x-space and its properties are determined. In Sec. 4, the instantaneous limit of the Green's function in momentum space is taken. In Sec. 5, the analyticity properties of the Green's function in the instantaneous limit are studied and the spectral functions are reconstructed. Sec. 6 is devoted to a brief application to the calculation of the quark condensate. Conclusion follows in Sec. 7. An appendix is devoted to the study of the properties of the threshold masses and to their determination.

2   Two-dimensional QCD in the large-N_c limit

On technical grounds, considerable simplification occurs in Eq. (1.1) in two-dimensional QCD at large N_c. Wilson loop averages are now exponential functionals of the areas of the surfaces lying inside the contours [13, 14]. First, for simple contours, made of junctions of simpler ones, the Wilson loop average factorizes into the product of the averages over the simpler contours. This feature leads to the disappearance in the kernels of Eq. (1.1) of nested-type diagrams. Second, crossed type diagrams as well as quark loop contributions disappear. Third, the second-order functional derivative of the logarithm of the Wilson loop average is a (two-dimensional) delta-function. As a consequence, since each derivation acts on a different segment, all functional derivatives of higher-order than the second vanish. The series of kernels in the right-hand side of Eq. (1.1) then reduces to its first term. Equation (1.1) becomes:

$$(i\gamma.\partial_{(x)} - m)S(x-x') = i\delta^2(x-x') - i\gamma^\mu \left(\frac{\bar{\delta}^2 F_3(x',x,y)}{\bar{\delta}x^{\mu-}\bar{\delta}y^{\alpha+}} \right) S(x-y)\,\gamma^\alpha\,S_{(2)}(y,x';x), \quad (2.1)$$

where F_3 is the logarithm of the Wilson loop average along the triangle xyx'. The functional derivatives acting on F_3 represent integrated derivatives on the segments xx' and yx', x' remaining fixed [4]; one has:

$$\frac{\bar{\delta}^2 F_3(x', x, y)}{\bar{\delta}x^\mu - \bar{\delta}y^{\alpha+}} = 2i\sigma(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})(x' - x)^\nu(y - x')^\beta$$

$$\times \int_0^1 d\lambda\, d\lambda'\, \lambda\, \lambda'\delta^2(\lambda'(y - x') + \lambda(x' - x)), \qquad (2.2)$$

where σ is the string tension. The δ-function forces the point y to lie on the line $x'x$. In the present theory, the 2PGIQGF $S_{(2)}$ is related in a simple way to S: $S_{(2)}(y, x'; x) = e^{F_3(x',x,y)}S(y - x')$. Integrating with respect to y and one of the λs with an integration by parts, one obtains the following (exact) equation:

$$(i\gamma.\partial - m)S(x) = i\delta^2(x) - \sigma\gamma^\mu(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})x^\nu x^\beta$$

$$\times \left[\int_0^1 d\lambda\, \lambda^2\, S((1 - \lambda)x)\gamma^\alpha S(\lambda x) + \int_1^\infty d\xi\, S((1 - \xi)x)\gamma^\alpha S(\xi x)\right]. \qquad (2.3)$$

The first integral represents contributions when y is lying inside the segment $x'x$, while the second one contributions from outside the segment $x'x$ in the direction $x'x$.

3   The quark Green's function

Let us first observe that attempting to develop Eq. (2.3) into an iterative series with respect to the free propagator would lead to a series of terms with increasing singularities at the quark mass m. Replacing in the interaction dependent part of Eq. (2.3) S by the free propagator S_0, produces a term that behaves at the quark mass threshold as $(p^2 - m^2)^{-5/2}$, which, in turn leads to stronger singularities at higher orders. Therefore, the existence of a nonperturbative solution, with possibly stable properties concerning its singularities, would represent the only issue to have access to the structure of the complete solution. It turns out that Eq. (2.3) does have such a solution satisfying the required boundary conditions and expressible in analytic form. A complete understanding of the spectral properties of the Green's function S requires an analysis in momentume space. We defer that study to Secs. 4 and 5. In this section, we shall display the solution and shall check in x-space that it satisfies Eq. (2.3).

Decomposing the momentum space 2PGIQGF $S(p)$ along vector and scalar invariants F_1 and F_0,

$$S(p) = \gamma.pF_1(p^2) + F_0(p^2), \qquad (3.1)$$

the solutions for F_1 and F_0, for complex p^2, are:

$$F_1(p^2) = -i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n \frac{1}{(M_n^2 - p^2)^{3/2}}, \qquad (3.2)$$

$$F_0(p^2) = i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} (-1)^n b_n \frac{M_n}{(M_n^2 - p^2)^{3/2}}, \qquad (3.3)$$

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where the coefficients b_n are parametrized as

$$b_n = \frac{\sigma^2}{M_n + \overline{m}_n + (-1)^n m}. \tag{3.4}$$

The mass parameters M_n have real positive values, greater than the free quark mass m, increasing with n, while the parameters $\overline{m}_n$ generally have real positive values decreasing, for a fixed parity of the index n, with n. They are defined through an infinite set of algebraic equations which will be displayed below. Their asymptotic values, for large n, such that $n \gg m^2/(\sigma\pi)$, are:

$$M_n^2 \simeq \sigma\pi n, \qquad \overline{m}_n \simeq \frac{\sigma}{\pi}\frac{1}{M_n}. \tag{3.5}$$

The functions $(M_n^2 - p^2)^{3/2}$ are defined with cuts starting from their branch points and going to $+\infty$ on the real axis; they are real below their branch points down to $-\infty$ on the real axis.

From Eqs. (3.2) and (3.3) we deduce that the singularities of the functions F_1 and F_0 are located on the positive real axis of p^2 and represented by an infinite number of branch points or thresholds with stronger singularities than simple poles. That feature is an indication that the resulting singularities are not of the observable type.

The first threshold is located at M_1^2, which is positive and larger than the free quark mass squared m^2. For massless quarks, M_1 remains positive.

To analyze the solution in x-space, we define the Fourier transform of the decomposition (3.1):

$$S(x) = i\gamma.\partial F_1(r) + F_0(r), \qquad r = \sqrt{-x^2}. \tag{3.6}$$

We introduce modified functions $\widetilde{F}_1(r)$ and $\widetilde{F}_0(r)$ such that

$$S(x) = \frac{1}{2\pi}\left(\frac{i\gamma.x}{r}\widetilde{F}_1(r) + \widetilde{F}_0(r)\right), \quad \widetilde{F}_1(r) = -2\pi\frac{\partial F_1(r)}{\partial r}, \quad \widetilde{F}_0(r) = 2\pi F_0(r). \tag{3.7}$$

From the Fourier transforms of Eqs. (3.2) and (3.3), one obtains:

$$\widetilde{F}_1(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n\, e^{-M_n r}, \qquad \widetilde{F}_0(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} (-1)^{n+1} b_n\, e^{-M_n r}. \tag{3.8}$$

In terms of the functions $\widetilde{F}_1$ and $\widetilde{F}_0$, Eq. (2.3) decomposes into two equations (here written for $r \neq 0$):

$$\frac{\partial}{\partial r}(r\widetilde{F}_1(r)) + mr\widetilde{F}_0(r) = \frac{\sigma r^3}{2\pi}\Bigg\{ \int_0^1 d\lambda\,\lambda^2\left[\widetilde{F}_1((1-\lambda)r)\widetilde{F}_1(\lambda r) - \widetilde{F}_0((1-\lambda)r)\widetilde{F}_0(\lambda r)\right]$$
$$- \int_1^\infty d\xi\left[\widetilde{F}_1((\xi-1)r)\widetilde{F}_1(\xi r) + \widetilde{F}_0((\xi-1)r)\widetilde{F}_0(\xi r)\right]\Bigg\}, \tag{3.9}$$

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$$\frac{\partial}{\partial r}(\widetilde{F}_0(r)) + m\widetilde{F}_1(r) = \frac{\sigma r^2}{2\pi}\Big\{ \int_0^1 d\lambda\, \lambda^2 \left[\widetilde{F}_1((1-\lambda)r)\widetilde{F}_0(\lambda r) - \widetilde{F}_0((1-\lambda)r)\widetilde{F}_1(\lambda r)\right]$$
$$- \int_1^\infty d\xi \left[\widetilde{F}_1((\xi-1)r)\widetilde{F}_0(\xi r) + \widetilde{F}_0((\xi-1)r)\widetilde{F}_1(\xi r)\right] \Big\}.$$

(3.10)

Replacing in them $\widetilde{F}_1$ and $\widetilde{F}_0$ with their expressions (3.8) and considering each exponential function with a given M_n as an independent function for $r \neq 0$, one obtains the following three sets of consistency conditions, valid for all $n = 1, \ldots, \infty$, n not summed:

$$M_n = (-1)^{n+1}m + \frac{1}{2}\sum_{q \neq n}(1-(-1)^{n+q})\frac{b_q}{(M_n - M_q)^2}, \tag{3.11}$$

$$1 = -\sum_{q \neq n}(1-(-1)^{n+q})\frac{b_q}{(M_n - M_q)^3}, \tag{3.12}$$

$$\sum_{q \neq n}(1-(-1)^{n+q})\frac{b_q}{(M_n - M_q)} + \sum_q (1+(-1)^{n+q})\frac{b_q}{(M_n + M_q)} = 0. \tag{3.13}$$

Equations (3.11) and (3.13) are consequences of both of Eqs. (3.9) and (3.10), while Eq. (3.12) results from Eq. (3.9) only; its components play a particular role in the limit $r \to 0$. In that limit, the right-hand side of Eq. (3.12), which is inserted in the series of the exponential functions, gives a vanishing contribution for antisymmetry reasons, while its left-hand side contributes globally as a divergent constant reproducing the delta-function of the right-hand side of Eq. (2.3). Equations (3.11)-(3.13) determine completely the mass parameters M_n and $\overline{m}_n$, up to a global normalization factor fixed by the inhomogeneous part of Eq. (2.3). Details of the resolution method are presented in the appendix. Equations (3.11) and (3.12) mainly determine the parameters M. The parameters $\overline{m}$ are asymptotically negligible in front of the Ms and they play a secondary role in the above equations. They are mainly determined through Eqs. (3.13). From the latter equations and from Eqs. (3.11), one derives the following ones:

$$\overline{m}_n = \frac{1}{2}\sum_{q=1}^\infty (1+(-1)^{n+q})\frac{b_q}{(M_n + M_q)^2}, \qquad n = 1, 2, \ldots, \tag{3.14}$$

which are more appropriate for an iterative determination of the $\overline{m}$s.

Finally, we study the effect of the inhomogeneous part of Eq. (2.3). From Eqs. (3.9) and (3.10) one finds that the interaction dependent parts, represented by the right-hand sides, vanish in the limit $r = 0$. In momentum space this means that the asymptotic behavior of $S(p)$ is governed, for spacelike momenta, by the free propagator, as can be deduced from the inhomogeneous part of Eq. (2.3). The asymptotic behavior of the scalar function F_0, however, crucially depends on whether the quark is massless or not [15].

We first check the behavior of F_1 from Eq. (3.2). Taking $|p^2|$ large and neglecting in the right-hand side the M_n^2s in front of p^2, one finds a divergent coefficient which is

an indication that the asymptotic behavior of F_1 is slower than $(-p^2)^{-3/2}$. To evaluate the asymptotic tail of the series, one can replace b_n by its asymptotic behavior $\sigma^2/M_n \simeq \sigma^2/\sqrt{\sigma \pi n}$ and convert the series into an integral. One then obtains for the asymptotic behavior

$$F_1(p^2) \underset{p^2 \to -\infty}{=} \frac{i}{p^2}, \tag{3.15}$$

in agreement with what is expected from the inhomogeneous part of Eq. (2.3). In x-space, $F_1(r)$ behaves for spacelike x and small r as $(1/(4\pi)) \ln(1/(\sigma r^2))$ [16] and $\widetilde{F}_1(r)$ as $1/r$.

For the study of the asymptotic behavior of F_0, for $m \neq 0$, one again sticks to the asymptotic tail of the series which provides the dominant behavior. For this, one combines in the corresponding alternating series two successive terms, which leads to the factorisation of the parameter m. Proceeding then as for F_1, one finds, as expected,

$$F_0(p^2) \underset{p^2 \to -\infty}{=} \frac{im}{p^2}, \qquad m \neq 0. \tag{3.16}$$

In x-space, $F_0(r)$ behaves for spacelike x and small r as $(m/(4\pi)) \ln(1/(\sigma r^2))$.

When $m = 0$, the mass dependent terms in the left-hand side of Eqs. (3.9) and (3.10) disappear. Eq. (3.10) implies that $\frac{\partial}{\partial r}(\widetilde{F}_0(r))$ vanishes at $r = 0$, which leads to a constraint on the masses and residues:

$$\sum_{n=1}^{\infty} (-1)^n b_n M_n = 0, \qquad m = 0. \tag{3.17}$$

$F_0(0)$ is then a constant. Analyzing the behavior of the right hand-side at small r, one finds for $F_0(r)$ the following behavior: $F_0(r) = F_0(0)\left[1 - (\sigma r^2/(4\pi)) \ln(1/(\sigma r^2))\right] + O(r^2)$. By Fourier transformation to momentum space, the first term produces a delta-function that does not contribute to the asymptotic behavior; it is the logarithmic term that provides the dominant term in the asymptotic region [16] ($r^2 = -x^2$, Eq. (3.6)):

$$F_0(p^2) \underset{p^2 \to -\infty}{=} -\frac{4i\sigma F_0(x=0)}{(p^2)^2} = \frac{2i\sigma}{N_c} \frac{\langle \overline{\psi}\psi \rangle}{(p^2)^2}, \qquad m = 0, \tag{3.18}$$

where in the second equation we have introduced the one-flavor quark condensate $\langle \overline{\psi}(0)\psi(0) \rangle = -2N_c F_0(x=0)$.

We have thus shown that expressions (3.1)-(3.3), or equivalently (3.6)-(3.8), are solutions of Eq. (2.3), provided the constraints (3.11)-(3.13) are satisfied.

On practical grounds, there remains to solve the latter equations for the parameters M_n and $\overline{m}_n$ ($n = 1, 2, \ldots$). The properties of those equations are studied in the appendix, where also details about their numerical resolution are presented. We have calculated, for several values of the quark mass, $m = 0., 0.1, 1., 5., 20.$, in mass unit of $\sqrt{\sigma/\pi} \simeq 240$ MeV, the first 900 values of the parameters M_n and $\overline{m}_n$, with better accuracy than 10^{-4}

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for the first 400 values. For massless or light quarks, the asymptotic values (3.5) are reached within a few per cent accuracy after a few dozens of ns. With increasing quark masses, the asymptotic limit is reached more slowly. Approximate formulas for the next-to-leading terms in the asymptotic behavior of M_n are presented in the appendix. We display in Tables 1 and 2 a few typical values of M and $\overline{m}$.

n	1	2	3	4	5	6	225	226
$m = 0$	1.161	3.043	4.301	5.287	6.127	6.868	46.96	47.06
$m = 0.1$	1.248	3.103	4.344	5.323	6.158	6.897	46.96	47.06
$m = 1$	2.053	3.706	4.823	5.745	6.536	7.246	47.03	47.14
$m = 5$	5.833	7.059	7.898	8.642	9.287	9.887	47.92	48.02
$m = 20$	20.59	21.41	21.97	22.48	22.93	23.36	55.61	55.70

Table 1: A few values of M, in mass unit of $\sqrt{\sigma/\pi}$, for several values of the quark mass m.

n	1	2	3	4	5	6	225	226
$m = 0$	1.052	0.217	0.282	0.147	0.191	0.120	0.022	0.021
$m = 0.1$	0.969	0.207	0.282	0.143	0.192	0.117	0.022	0.021
$m = 1$	0.553	0.145	0.257	0.110	0.189	0.094	0.023	0.019
$m = 5$	0.179	0.056	0.143	0.051	0.125	0.048	0.025	0.016
$m = 20$	0.049	0.016	0.047	0.016	0.046	0.016	0.022	0.010

Table 2: A few values of $\overline{m}$, in mass unit of $\sqrt{\sigma/\pi}$, for several values of the quark mass m.

We present in Fig. 1 the function iF_0 for spacelike p and in Fig. 2 its real part for timelike p, and similarly for the function iF_1 in Figs. 3 and 4, for $m = 0$. Their imaginary parts (the spectral functions multiplied with π) have behaviors that resemble those of the real parts in the timelike region, except that they are null below M_1 and that the infinite singularities begin just from the right of the Ms. A relationship between $iF_1(0)$ and $iF_0(0)$ exists; it is presented in the appendix.

The x-space functions $F_0(r)$ and $F_1(r)$ [Eqs. (3.7)-(3.8)] are positive and monotone decreasing functions of r for spacelike x. For timelike x, the two functions are exponentially oscillating.
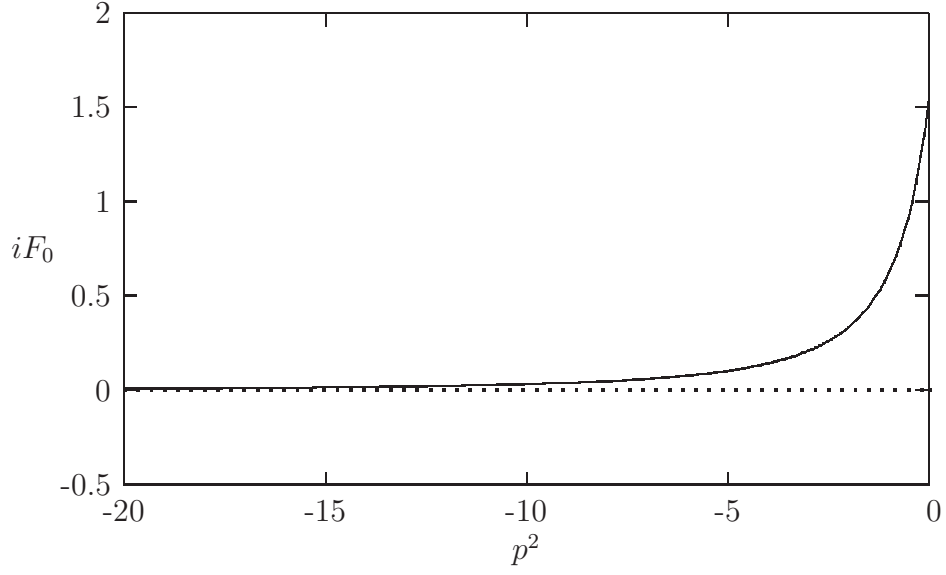
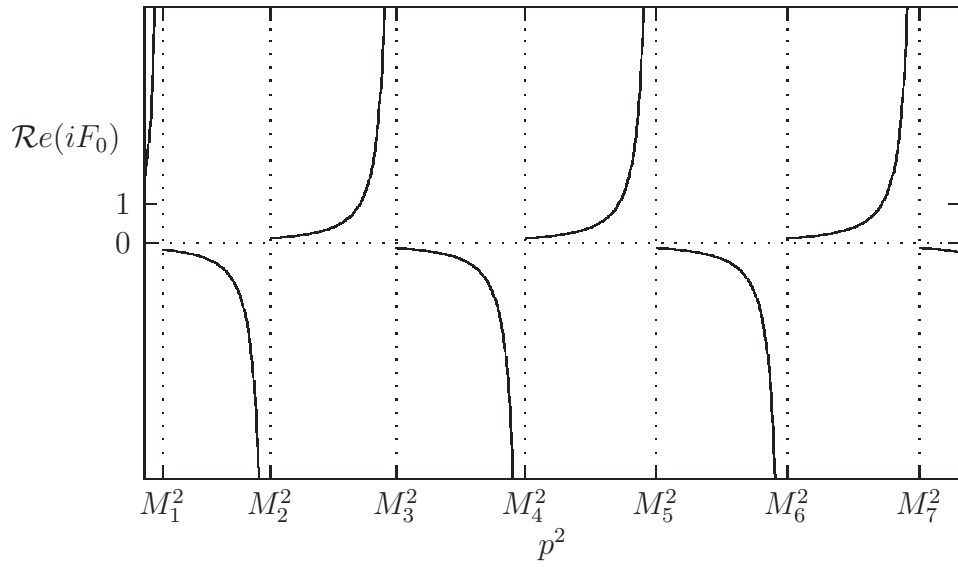Figure 1: The function iF_0 for spacelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.

Figure 2: The real part of the function iF_0 for timelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.
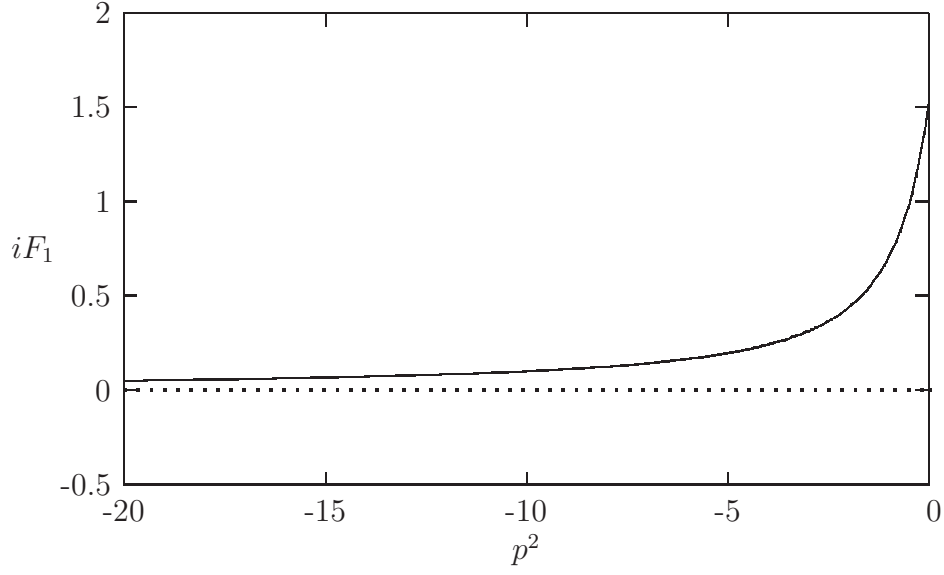
Figure 3: The function iF_1 for spacelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.
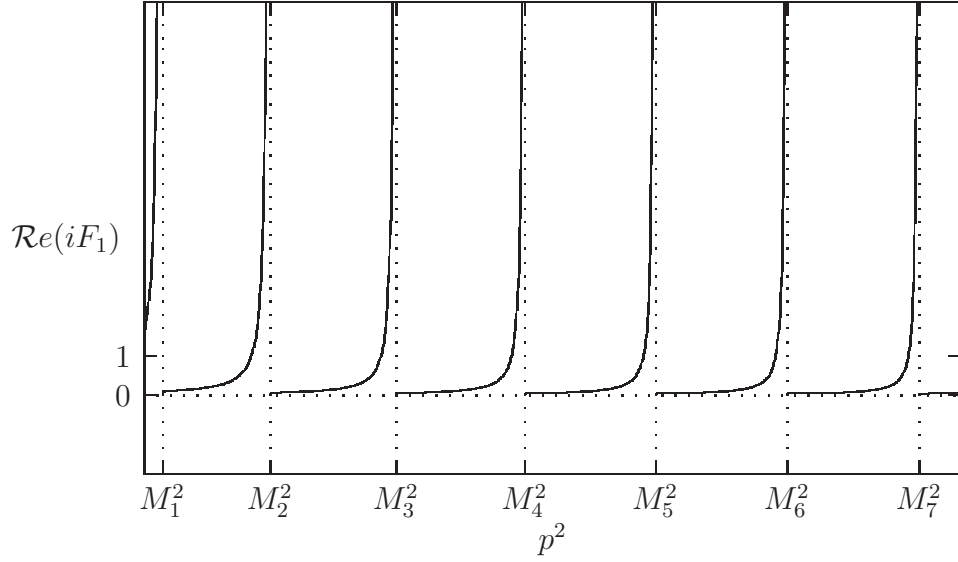

Figure 4: The real part of the function iF_1 for timelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.

4   Momentum space analysis: the instantaneous limit

A direct resolution of Eq. (2.3) in momentum space is complicated because of the presence of the additional integrations upon the variables λ and ξ. Nevertheless, a two-step resolution, using the analyticity properties resulting from the presence of spectral functions, is possible. This section and the following one describe the corresponding procedure.

The starting point is based on the observation that Eq. (2.3) is quasi-local in x, in the sense that in the right-hand side of the equation x undergoes only dilatation operations. Therefore, projections of the Green's function on particular subspaces still give rise to the same equation (for $x \neq 0$). Of particular interest is the instantaneous limit $x^0 = 0$. In that case, the variable r of Eq. (3.6) simply becomes the modulus of the space component x^1: $r = |x^1|$. The functions $\widetilde{F}_1(r)$ and $\widetilde{F}_0(r)$, introduced in Eqs. (3.7), keep the same definitions and Eqs. (3.9) and (3.10) remain unchanged in form with the only modification that r has become a one-dimensional variable.

We pass now to momentum space with one-dimensional Fourier transformation. The momentum variable k will designate the modulus of the space component p_1 of the two-dimensional vector p: $k = |p_1|$. The Fourier transform of $\widetilde{F}_0(r)$ will be defined with the cosine function, while that of $\widetilde{F}_1(r)$, which is defined as a derivative of $F_1(r)$ with respect to r [Eq. (3.7)], with the sine function. We have the definitions

$$f_0(k) = 2\int_0^\infty dr\,\cos(kr)\,\widetilde{F}_0(r), \quad f_1(k) = 2\int_0^\infty dr\,\sin(kr)\,\widetilde{F}_1(r), \quad r = |x^1|, \quad (4.1)$$

$$F_{0I}(k) = \int_{-\infty}^\infty \frac{dp_0}{2\pi}\,F_0(p) = \frac{f_0(k)}{2\pi}, \quad F_{1I}(k) = \int_{-\infty}^\infty \frac{dp_0}{2\pi}\,F_1(p) = \frac{f_1(k)}{2\pi k}, \quad k = |p_1|. \quad (4.2)$$

Replacement of $\widetilde{F}_1(r)$ and $\widetilde{F}_0(r)$ in Eqs. (3.9) and (3.10) by their one-dimensional Fourier transforms allows the integration with respect to the variables λ and ξ. One ends up with the equivalent equations for $f_1(k)$ and $f_0(k)$:

$$
\begin{aligned}
k\frac{\partial f_1(k)}{\partial k} + m\frac{\partial f_0(k)}{\partial k} =&\ -\frac{\sigma}{2\pi^2}\int_0^\infty dq\,\Big\{ (f_1^2(q) + f_0^2(q))\,\Big(\frac{1}{(q+k)^3} - \frac{1}{(q-k)^3}\Big) \\
&\ -(f_1(q)\frac{\partial f_1(k)}{\partial k} - f_0(q)\frac{\partial f_0(k)}{\partial k})\,\frac{1}{(q+k)^2} \\
&\ +(f_1(q)\frac{\partial f_1(k)}{\partial k} + f_0(q)\frac{\partial f_0(k)}{\partial k})\,\frac{1}{(q-k)^2} \Big\} - \frac{\sigma}{\pi^2}f_1^2(0)\frac{\partial}{\partial k}\delta(k),
\end{aligned}
$$
$$(4.3)$$

$$
\begin{aligned}
kf_0(k) - mf_1(k) =&\ \frac{\sigma}{2\pi^2}\int_0^\infty dq\,\Big\{ (f_1(q)f_0(k) + f_0(q)f_1(k))\,\frac{1}{(q+k)^2} \\
&\ -(f_1(q)f_0(k) - f_0(q)f_1(k))\,\frac{1}{(q-k)^2} \Big\}.
\end{aligned}
$$
$$(4.4)$$

Equations (4.3) and (4.4) display singular kernels. Their effects on the solutions can be studied by expanding the variable q around k in the functions $f_1(q)$ and $f_0(q)$. It turns out that Eq. (4.4) is finite, while Eq. (4.3) is in general divergent. Finite solutions exist, provided that they satisfy the constraint $f_1^2 + f_0^2 =$ constant. The value of the constant can be determined from the inhomogeneous part of Eq. (2.3) and the property that f_1 and f_0 should behave asymptotically as in a free field theory, i.e., $f_1(k) \to \pi$ and $f_0(k) \to 0$ for $k \to \infty$. This fixes the constant to π^2:

$$f_1^2(k) + f_0^2(k) = \pi^2. \tag{4.5}$$

Parametrizing the functions f_1 and f_0 by means of an angle $\varphi(k)$ as

$$f_1(k) = \pi \cos \varphi(k), \qquad f_0(k) = \pi \sin \varphi(k), \tag{4.6}$$

Eq. (4.3) reproduces Eq. (4.4) in addition to its last term (the derivative of the delta-function). The compatibility of the two equations then requires the vanishing of the coefficient of the latter term, i.e., of $f_1(0)$:

$$f_1(0) = 0, \quad \text{or,} \quad \varphi(0) = \frac{\pi}{2}. \tag{4.7}$$

The final equation is then:

$$
\begin{aligned}
k \sin \varphi(k) - m \cos \varphi(k) \;=\; & \frac{\sigma}{2\pi} \Big\{ \cos \varphi(k) \int_0^\infty dq \, \sin \varphi(q) \, \big(\frac{1}{(q+k)^2} + \frac{1}{(q-k)^2}\big) \\
& + \sin \varphi(k) \int_0^\infty dq \, \cos \varphi(q) \, \big(\frac{1}{(q+k)^2} - \frac{1}{(q-k)^2}\big) \Big\}.
\end{aligned}
\tag{4.8}
$$

Let us emphasize at this point that the boundary condition (4.7), which is a necessary condition for the existence of solutions to Eqs. (4.3)-(4.4), eliminates the free case $\varphi = 0$ when $m = 0$ from the set of possibilities, which otherwise would fit Eq. (4.8). This conclusion can also be reached directly in x-space, by inspecting Eq. (2.3).

Equation (4.8) is identical to the one satisfied by the ordinary quark propagator in its instantaneous limit in the axial gauge [17, 18, 19]. Therefore, the 2PGIQGF and the ordinary quark propagator in the axial gauge coincide in their instantaneous limits.

More generally, Eq. (4.8) is analogous to the self-energy equation obtained in the instantaneous limit in the four-dimensional theory, using Coulomb gluons for the description of the confinement mechanism [20, 21, 22, 23, 24]. Here, however, as well as in the axial gauge of the two-dimensional theory, the ordinary quark propagator is a divergent quantity due to the infra-red singularity of the gluon-propagator, and is made finite with the aid of a regularization procedure. It turns out that its instantaneous limit is free of

divergences and satisfies a finite equation [22]. This is the reason why the interface between the 2PGIQGF, which in general is expected to be an infra-red finite quantity, and the ordinary quark propagator in a non-covariant gauge, exists only in the instantaneous limit.

From Eq. (4.8), one deduces that asymptotically, as $k \to \infty$, $\sin \varphi$ behaves as m/k when $m \neq 0$ and as $\frac{2\sigma}{k^3} \int_0^\infty \frac{dq}{2\pi} \sin \varphi(q)$ when $m = 0$, in agreement with the results obtained in Eqs. (3.16) and (3.18).

Equation (4.8) can be solved with the same methods as its four-dimensional analogue [22, 25]. The solution for $\sin \varphi$, for massless quarks, is presented in Fig. 5.
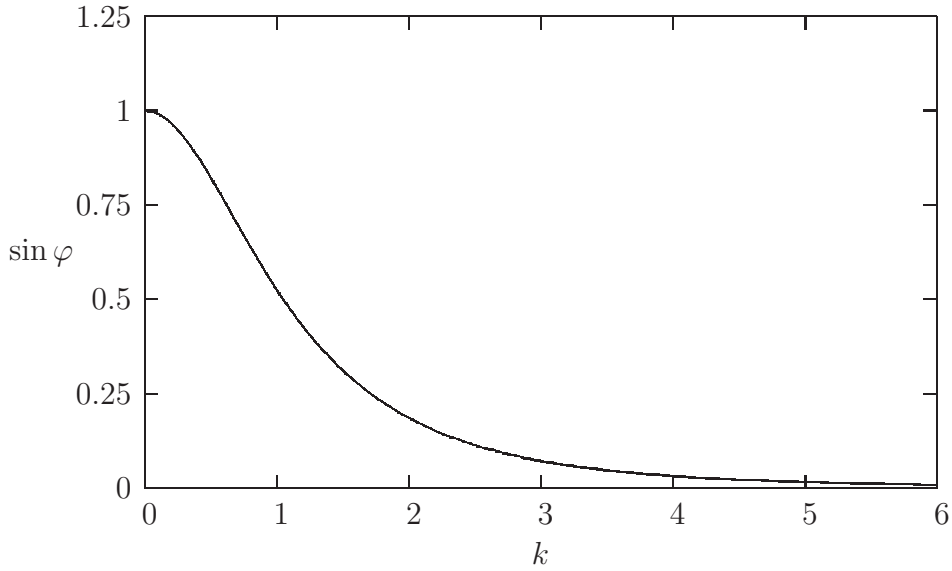

Figure 5: The function $\sin \varphi$, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.

5   Analyticity properties in the instantaneous limit

The numerical knowledge of the instantaneous limit of the functions $F_1(p)$ and $F_0(p)$ does not allow an immediate reconstruction of the whole covariant functions and in particular of their spectral functions. Nevertheless, the existence of underlying analyticity properties allows us to reach that aim by means of an analytic continuation. This section is devoted to that procedure.

To simplify the presentation, we shall assume that the series of spectral functions (1.2) sum up, by means of integrations by parts, into the lowest-order denominator. If the spectral functions possess strong singularities that forbid the above operations, one

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can bring outside the integrals the appropriate number of derivation operators. The spectral representation of the functions F_1 and F_0 then becomes:

$$F_1(p^2) = i \int_0^\infty ds \frac{\rho_1(s)}{(p^2 - s + i\varepsilon)}, \qquad F_0(p^2) = i \int_0^\infty ds \frac{\rho_0(s)}{(p^2 - s + i\varepsilon)}. \tag{5.1}$$

Taking the instantaneous limit of the latter functions [Eqs. (4.2) and (4.6)], one finds

$$\cos \varphi(k) = k \int_0^\infty ds \frac{\rho_1(s)}{\sqrt{k^2 + s}}, \qquad \sin \varphi(k) = \int_0^\infty ds \frac{\rho_0(s)}{\sqrt{k^2 + s}}. \tag{5.2}$$

In principle, the above integral equations could be inverted with the successive uses of the Hankel transform of order zero and the inverse Laplace transform [26]. However, the latter transform necessitates knowledge of analytic expressions. Also, the direct calculation of the product of the two transforms gives a singular expression, since the Bessel function of order zero is an analytic function. These difficulties can, however, be circumvented with the use of an analytic continuation method.

The key observation is that because of the properties of the spectral functions, assumed to be concentrated on the positive real axis, the instantaneous limit functions $\cos \varphi$ and $\sin \varphi$ are themselves analytic functions of k in its whole complex plane with possible singularities located on the imaginary axis. We are then entitled to continue those functions to complex values of k and in particular to the imaginary axis.

Let us consider for definiteness the function $\sin \varphi$. One can express it as the Laplace transform of some other function, which will be denoted h_0:

$$\sin \varphi(k) = \int_0^\infty dy\, h_0(y) e^{-yk}. \tag{5.3}$$

The exponential function e^{-yk} can be expressed as the Hankel transform of order zero of the function $1/\sqrt{k^2 + s}$:

$$e^{-yk} = \frac{y}{2} \int_0^\infty ds \frac{1}{\sqrt{k^2 + s}} J_0(y\sqrt{s}), \tag{5.4}$$

where J_0 is the Bessel function of order zero. Replacing in Eq. (5.3) the exponential function by its expression (5.4), one arrives at the identification

$$\rho_0(s) = \frac{1}{2} \int_0^\infty dy\, h_0(y) y J_0(y\sqrt{s}). \tag{5.5}$$

Finally, using for the Bessel function the representation $J_0(z) = (1/\pi) \int_{-1}^{+1} dt\, e^{izt}/\sqrt{1 - t^2}$ and continuing Eq. (5.3) to complex values of k, one obtains the formulas

$$\begin{aligned}
\rho_0(s) &= \frac{1}{2i\pi} \int_0^{\sqrt{s}} du \frac{1}{\sqrt{s - u^2}} \frac{\partial}{\partial u} \Big(\sin \varphi(-iu) - \sin \varphi(iu) \Big) \\
&= \frac{2}{2i\pi} \frac{\partial}{\partial s} \int_0^{\sqrt{s}} du \sqrt{s - u^2} \frac{\partial}{\partial u} \Big(\sin \varphi(-iu) - \sin \varphi(iu) \Big).
\end{aligned} \tag{5.6}$$

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Assuming that the real part of $\sin \varphi$ is an even function on the imaginary axis and the imaginary part an odd function, Eq. (5.6) means that the spectral function ρ_0 is determined from the imaginary part of the function $\sin \varphi$ on the imaginary axis.

A similar calculation as above can be repeated for the function $\cos \varphi$. One obtains

$$
\begin{aligned}
\rho_1(s) &= \frac{1}{2\pi} \int_0^{\sqrt{s}} du \, \frac{1}{\sqrt{s-u^2}} \frac{\partial}{\partial u} \Big[\frac{1}{u} (\cos \varphi(-iu) + \cos \varphi(iu)) \Big] \\
&= \frac{2}{2\pi} \frac{\partial}{\partial s} \int_0^{\sqrt{s}} du \, \sqrt{s-u^2} \frac{\partial}{\partial u} \Big[\frac{1}{u} (\cos \varphi(-iu) + \cos \varphi(iu)) \Big].
\end{aligned} \tag{5.7}
$$

The spectral function ρ_1 is then determined from the real part of $\cos \varphi$ on the imaginary axis.

To have an elementary check of the consistency of Eqs. (5.6) and (5.7), let us assume that the spectral functions are null for values of s below a positive mass-squared M^2. Then, for values of k smaller than M, one has from Eqs. (5.2) the expansions $\sin \varphi(k) = 1 + ak^2 + O(k^4)$ and $\cos \varphi(k) = bk + O(k^3)$. The analytic continution to the imaginary axis gives: $\sin \varphi(\pm ik) = 1 - ak^2 + O(k^4)$, $\cos \varphi(\pm ik) = \pm ibk + O(k^3)$, from which one verifies, through Eqs. (5.6) and (5.7), that ρ_0 and ρ_1, as expected, are null in the considered domain.

The evaluation of expressions (5.6) and (5.7) necessitates the analytic continuation of the functions $\sin \varphi$ and $\cos \varphi$ to the imaginary axis of the complex variable k. This can be achieved by using the integral equation (4.8), in which the integrands $\sin \varphi(q)$ and $\cos \varphi(q)$, which lie on the positive real axis, will now be assumed numerically known from the resolution of the equation for real positive values of k. The validity of that continuation is justified by the fact that Eq. (4.8) is the instantaneous limit of covariant scalar equations relative to the functions F_1 and F_0. The singularities of the integrals arise from the terms $1/(q-k)^2$, which actually contribute, by combining the various terms, as a principal value integrand. (The corresponding definition results from the Fourier sine or cosine transforms of the terms of the one-dimensional limit of Eq. (2.3).) The latter has a well-defined meaning as a generalized function [16]. One has:

$$
P \frac{1}{(q-k)} = \frac{1}{q - (k \mp i\varepsilon)} \pm i\pi \delta(q-k). \tag{5.8}
$$

[P designates the principal value.] The first term in the right-hand side of Eq. (5.8) is the boundary value of an analytic function and can readily be continued to the complex plane. According to the sign in front of $i\varepsilon$, the continuation will be done to the lower half-plane (minus sign) or to the upper half-plane (plus sign). The delta-function can be integrated with respect to q (actually, preceded by an integration by parts) and yields a term proportional to $\frac{\partial}{\partial k} \varphi(k)$, which is an analytic function of k and can be continued to the complex plane.

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We shall mainly be interested by the limit of k to the imaginary axis (with a small positive real part) and therefore shall directly consider this case:

$$k \to \eta \pm ik_i, \quad \eta > 0, \ \eta \simeq 0, \quad k_i > 0. \tag{5.9}$$

[For simplicity, η will henceforth be omitted from the formulas.] With the above definitions, the continuation of Eq. (4.8) to the imaginary axis takes the form

$$\pm ik_i \sin \varphi(\pm ik_i) - m \cos \varphi(\pm ik_i) + \frac{\sigma}{2} \frac{\partial}{\partial k_i} \varphi(\pm ik_i) \mp i \frac{\sigma}{2} \varphi'(0) \sin \varphi(\pm ik_i)$$
$$= \cos \varphi(\pm ik_i) \, g_1(k_i) \mp i \sin \varphi(\pm ik_i) \, g_2(k_i), \tag{5.10}$$

where $\varphi'(0) = \frac{\partial \varphi(k)}{\partial k}\big|_{k=0}$, and g_1 and g_2 are the following integrals:

$$g_1(k_i) = \frac{\sigma}{\pi} \int_0^\infty dq \, \frac{(q^2 - k_i^2)}{(q^2 + k_i^2)^2} \sin \varphi(q), \tag{5.11}$$

$$g_2(k_i) = \frac{\sigma}{\pi} \int_0^\infty dq \, \arctan\left(\frac{k_i}{q}\right) \frac{\partial^2}{\partial q^2} \cos \varphi(q)$$
$$= \frac{\sigma}{2} \varphi'(0) + \frac{\sigma}{\pi} k_i \int_0^\infty dq \, \frac{1}{(q^2 + k_i^2)} \frac{\partial}{\partial q} \cos \varphi(q). \tag{5.12}$$

The first expression of g_2 is useful for small values of k_i, while the second one for large values of k_i; one has $g_2(\infty) = \frac{\sigma}{2} \varphi'(0)$. g_1 and g_2 are in general negative. In magnitude, $|g_1|$ is a monotone decreasing function from a finite value at $k_i = 0$ down to zero at infinity; $|g_2|$ is a monotone increasing function from zero at $k_i = 0$ up to $\sigma |\varphi'(0)|/2$ at infinity ($\varphi'(0)$ is negative).

The functions $g_1(k_i)$ and $g_2(k_i)$ being known from the resolution of Eq. (4.8), we find that the continuation of the latter equation to the imaginary axis has yielded a first-order differential equation in φ, which can be analyzed in turn.

Decomposing φ into a real and an imaginary part, $\varphi(k) = \varphi_r(k) + i\varphi_i(k)$, Eqs. (5.10) decompose into two real equations:

$$\pm k_i \sin \varphi_r(\pm ik_i) \cosh \varphi_i(\pm ik_i) + m \sin \varphi_r(\pm ik_i) \sinh \varphi_i(\pm ik_i)$$
$$+ \frac{\sigma}{2} \frac{\partial}{\partial k_i} \varphi_i(\pm ik_i) \mp \frac{\sigma}{2} \varphi'(0) \sin \varphi_r(\pm ik_i) \cosh \varphi_i(\pm ik_i)$$
$$= - \sin \varphi_r(\pm ik_i) \sinh \varphi_i(\pm ik_i) \, g_1(k_i) \mp \sin \varphi_r(\pm ik_i) \cosh \varphi_i(\pm ik_i) \, g_2(k_i),$$
$$\tag{5.13}$$

$$\mp k_i \cos \varphi_r(\pm ik_i) \sinh \varphi_i(\pm ik_i) - m \cos \varphi_r(\pm ik_i) \cosh \varphi_i(\pm ik_i)$$
$$+ \frac{\sigma}{2} \frac{\partial}{\partial k_i} \varphi_r(\pm ik_i) \pm \frac{\sigma}{2} \varphi'(0) \cos \varphi_r(\pm ik_i) \sinh \varphi_i(\pm ik_i)$$
$$= \cos \varphi_r(\pm ik_i) \cosh \varphi_i(\pm ik_i) \, g_1(k_i) \pm \cos \varphi_r(\pm ik_i) \sinh \varphi_i(\pm ik_i) \, g_2(k_i).$$
$$\tag{5.14}$$

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The boundary values to be imposed are $\varphi_r(0) = \pi/2$ and $\varphi_i(0) = 0$, i.e., $\varphi(0) = \pi/2$ [Eq. (4.7)].

The above equations can be solved starting from a domain containing the origin, to implement the boundary conditions. It is found that the singularities of the functions $\sin \varphi$ and $\cos \varphi$ on the imaginary axis are represented by an infinite set of simple poles with locations at $k = \pm i M_n$, $n = 1, 2, \ldots$. Considering for instance the positive imaginary axis, one can divide it into domains, each one lying between two successive poles, the first domain lying between the origin and the first pole. In the first domain, one has $\varphi_r = \pi/2$; in the second one, $\varphi_r = -\pi/2$; in the third one, $\varphi_r = \pi/2$ (modulo 2π); φ_r thus alternates between the constant values $\pm\pi/2$ when passing from one domain to the other.

In each domain, φ_i is a monotone increasing or decreasing function of k_i (taking into account the signs and values of the functions g_1 and g_2). In the first domain, the solution for φ_i is given by the equation

$$\sigma \arctan(\tanh(\frac{\varphi_i(k_i)}{2})) = \int_0^{k_i} dk_i'\Big(- k_i' + \frac{\sigma}{2}\varphi'(0) - g_2(k_i') - (m + g_1(k_i')) \tanh \varphi_i(k_i')\Big),$$
(5.15)

which can be solved by iteration. [We henceforth omit in the argument of φ_i the factor i.] The position of the singularity M_1 corresponds to that value of k_i for which the right-hand side reaches the value $\sigma\pi/4$; at that position, φ_i becomes logarithmically infinite. One has then the equation

$$\int_0^{M_1} dk_i'\Big(k_i' - \frac{\sigma}{2}\varphi'(0) + g_2(k_i') + (m + g_1(k_i')) \tanh \varphi_i(k_i')\Big) = \sigma\frac{\pi}{4}.$$
(5.16)

In the $(n+1)$th domain $(n \geq 1)$, $M_n < k_i < M_{n+1}$, the corresponding equations are:

$$\sigma \arctan(\tanh(\frac{\varphi_i(k_i)}{2})) = (-1)^n\Big\{\sigma\frac{\pi}{4}$$
$$+ \int_{M_n}^{k_i} dk_i'\Big(- k_i' + \frac{\sigma}{2}\varphi'(0) - g_2(k_i') - (m + g_1(k_i')) \tanh \varphi_i(k_i')\Big)\Big\},$$
(5.17)

$$\int_{M_n}^{M_{n+1}} dk_i'\Big(k_i' - \frac{\sigma}{2}\varphi'(0) + g_2(k_i') + (m + g_1(k_i')) \tanh \varphi_i(k_i')\Big) = \sigma\frac{\pi}{2}.$$
(5.18)

The residues of $\sin \varphi$ and $\cos \varphi$ at the position of their poles are found by expanding in the various expressions k_i around M_n. One finds:

$$\sin \varphi(\pm i k_i) \underset{k_i \simeq M_n}{\sim} (-1)^{(n+1)}\frac{\sigma}{2} \frac{1}{(M_n + \overline{m}_n + (-1)^n m)} \frac{1}{(M_n - k_i \pm i\varepsilon)},$$
(5.19)

$$\cos \varphi(\pm i k_i) \underset{k_i \simeq M_n}{\sim} \pm i\frac{\sigma}{2} \frac{1}{(M_n + \overline{m}_n + (-1)^n m)} \frac{1}{(M_n - k_i \pm i\varepsilon)},$$
(5.20)

$$\overline{m}_n = -\frac{\sigma}{2}\varphi'(0) + g_2(M_n) + (-1)^n g_1(M_n).$$
(5.21)

The real parts of $\sin\varphi$ and $i\cos\varphi$ for k on the positive imaginary axis are shown in Figs. 6 and 7. They are even and odd functions of k_i, respectively.
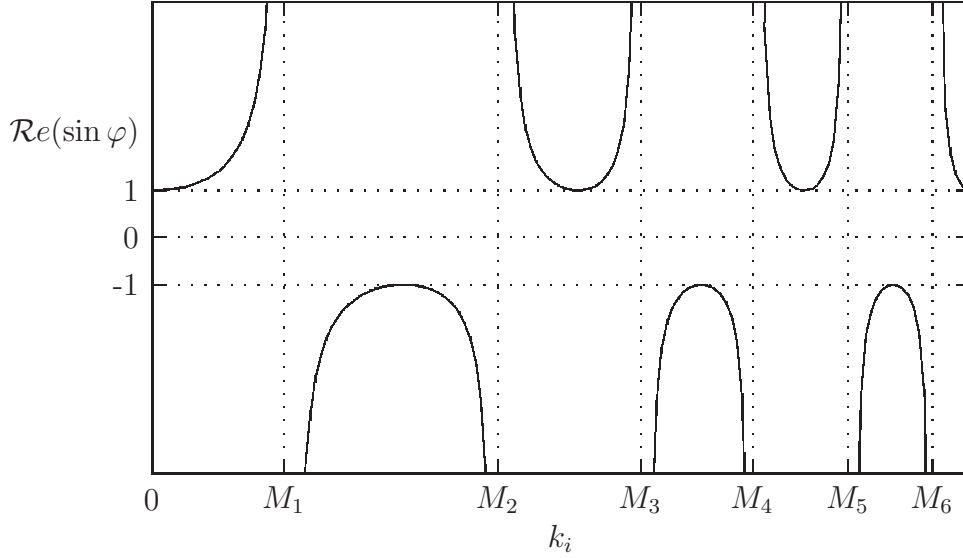

Figure 6: The real part of the function $\sin\varphi$ on the positive imaginary axis.

The spectral functions ρ_0 and ρ_1 are obtained from Eqs. (5.6) and (5.7). One finds:

$$\rho_0(s) = \frac{\sigma}{2}\sum_{n=1}^{\infty}(-1)^n\frac{M_n}{(M_n+\overline{m}_n+(-1)^nm)}\frac{\theta(s-M_n^2)}{(s-M_n^2)^{3/2}}, \tag{5.22}$$

$$\rho_1(s) = -\frac{\sigma}{2}\sum_{n=1}^{\infty}\frac{1}{(M_n+\overline{m}_n+(-1)^nm)}\frac{\theta(s-M_n^2)}{(s-M_n^2)^{3/2}}. \tag{5.23}$$

The functions $\sin\varphi$ and $\cos\varphi$ on the positive real axis are reconstituted from Eqs. (5.2). [Since the ρs have strong singularities at the various thresholds, one has to use for them the first version of equations (5.6) and (5.7) and make an interchange in the order of integrations between u and s.] One obtains:

$$\sin\varphi(k) = -\sigma\sum_{n=1}^{\infty}(-1)^n\frac{M_n}{(M_n+\overline{m}_n+(-1)^nm)}\frac{1}{(k^2+M_n^2)}, \tag{5.24}$$

$$\cos\varphi(k) = \sigma k\sum_{n=1}^{\infty}\frac{1}{(M_n+\overline{m}_n+(-1)^nm)}\frac{1}{(k^2+M_n^2)}. \tag{5.25}$$

As a check, one verifies that the curve of $\sin\varphi(k)$ obtained from Eq. (5.24) with the numerical values of the mass parameters M_n and $\overline{m}_n$ coincides with that obtained from the direct resolution of Eq. (4.8) [Fig. 5].

One can go further, by transforming the informations coming from the instantaneous limit into an infinite set algebraic equations. First, going back to the defining equations
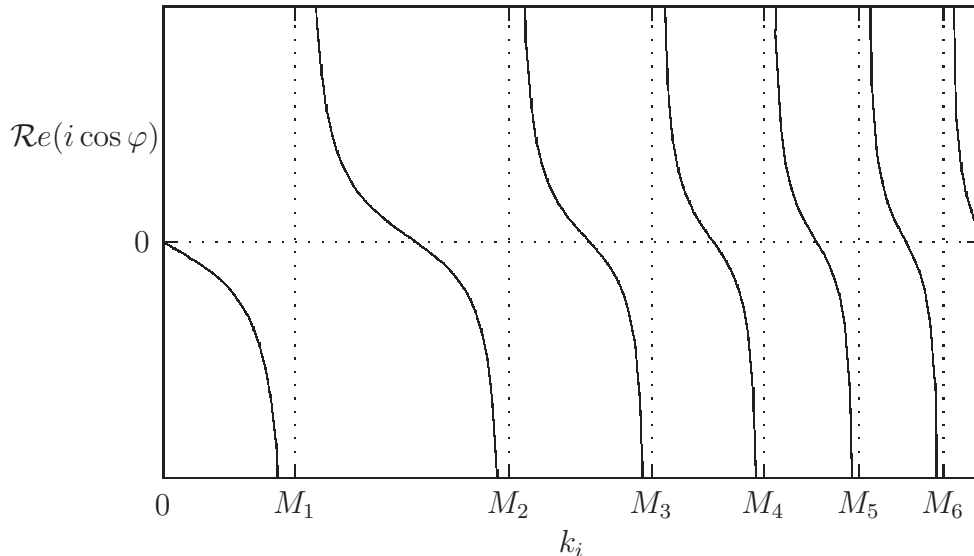
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Figure 7: The real part of the function $i \cos \varphi$ on the positive imaginary axis.

of the parameters $\overline{m}_n$ [Eqs. (5.21)] and calculating now the functions g_1 and g_2 with the use of the expressions (5.24) and (5.25) of $\sin \varphi$ and $\cos \varphi$ on the real axis, one finds the set of equations (3.14) allowing the calculation of the $\overline{m}$s for known Ms.

Second, using expressions (5.24) and (5.25) in Eq. (4.8), one finds, as consistency conditions Eqs. (3.11) and (3.13). Equation (3.12) is implicit here, since odd-index and even-index parameters succeed to each other. [Their definition is not a mere artifact, their defining equations being different, cf. Eqs. (5.17) and (5.21).] The equation $\sin^2 \varphi + \cos^2 \varphi = 1$, which is a non-trivial constraint for the expressions (5.24)-(5.25), leads also to the previous consistency relations. One thus finds at the end the same algebraic equations as those found in the x-space analysis.

Finally, the expressions (5.23) and (5.22) of the spectral functions allow us to reconstruct the completely covariant functions F_1 and F_0, given in Eqs. (3.2) and (3.3).

This completes the study and determination of the quark Green's function in momentum space.

6 The quark condensate

A quantity of interest is the quark condensate, which, in the one-flavor case and for massless quarks, has the expression [Eq. (3.6)]

$$\langle \overline{\psi}(0)\psi(0) \rangle = -\text{tr}_{c,sp} S(m=0, x)\big|_{x=0} = -2N_c F_0(m=0, x=0), \qquad (6.1)$$

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$\text{tr}_{c,sp}$ meaning the trace operation taken on color and spinor indices. This quantity, being the vacuum average of a local gauge invariant operator, is also calculable from the ordinary quark propagator and does not need for its evaluation the recourse to gauge invariant Green's functions. Nevertheless, it is interesting to compare, as a check, its value obtained in the present approach to that obtained in previous approaches.

The quark condensate is calculated from the series expression of the scalar function $F_0(x)$ [Eqs. (3.7), (3.8) and (3.4)] for $m = 0$:

$$\langle \overline{\psi}(0)\psi(0) \rangle = \frac{2N_c\sigma}{4} \sum_{n=1}^{\infty} (-1)^n \frac{1}{(M_n + \overline{m}_n)} \simeq -293 \text{ MeV}, \qquad (6.2)$$

the numerical value being obtained with $N_c = 3$ and $\sqrt{\sigma} = 425$ MeV. [The asymptotic tail of the series is evaluated using the asymptotic expression of M_n and neglecting the $\overline{m}$s.]

The above numerical value is in agreement with that obtained in the axial gauge from the integral of the function $\sin \varphi$ for massless quarks [18] ($\langle \overline{\psi}\psi \rangle \simeq -295$ MeV) and with that obtained in the light-cone gauge using operator product expansion methods [27, 28] ($\langle \overline{\psi}\psi \rangle = -N_c\sqrt{\sigma/(6\pi)} \simeq -294$ MeV).

7 Conclusion

The study of the gauge invariant quark Green's function in two-dimensional QCD in the large-N_c limit by means of the integrodifferential equation it satisfies has allowed us to determine the properties of its spectral functions. Those functions directly probe the contributions of quarks and gluons in a sector where no color-singlet states can contribute. The main results can be summarized as follows.

1) The spectral functions are infra-red finite and lie on the positive real axis of the variable p^2 in Minkowski space (p timelike). No singularities in the complex plane or on the negative real axis have been found. This feature underlines the fact that quarks and gluons contribute to the spectral functions like physical particles with positive energies.

2) The singularities of the spectral functions are represented by an infinite number of threshold singularities, characterized by positive real masses M_n ($n = 1, 2, \ldots$) extending from a lowest positive value up to infinity. The corresponding singularities near the thresholds are of the type $(p^2 - M_n^2)^{-3/2}$ and therefore are stronger than simple pole singularities, a feature which reflects the difficulty in observation of quarks as free particles.

3) The threshold masses M_n remain positive in the limit of massless quarks and maintain the scalar part of the Green's function at a non-zero value.

The present method of investigation can also be applied to four-dimensional QCD, in which case, however, several difficulties arise. First, on practical grounds, the inte-

grodifferential equation of the gauge invariant Green's function can no longer be solved in an exact way, because of the presence of an infinite number of kernels and of the different contributions of short- and large-distance interactions. Second, the presence of short-distance interactions induces divergences which necessitate an appropriate treatment through renormalization. Third, the Wilson loop averages are no longer known exactly.

Another domain of investigation is the four-point gauge invariant Green's function, which is naturally related to the bound state equation of mesons.

Acknowledgements

A   Determination of the spectral function parameters

This appendix is devoted to the determination of the threshold masses and factors that appear in the spectral functions. In the first part, we study the mathematical properties of the corresponding equations; in the second part, the asymptotic behaviors of the mass parameters M_n and $\overline{m}_n$ for large n are found and in the third part, details of the numerical resolution of the equations are given.

A.1   Mathematical properties

We notice that Eqs. (3.11) and (3.12) separate into two complementary sets according to the parity of the index n:

$$M_{2n+1} = m + \sum_{q=0}^{\infty} \frac{b_{2q+2}}{(M_{2n+1} - M_{2q+2})^2}, \quad n = 0, 1, 2, \ldots, \tag{A.1}$$

$$1 = -2 \sum_{q=0}^{\infty} \frac{b_{2q+2}}{(M_{2n+1} - M_{2q+2})^3}, \quad n = 0, 1, 2, \ldots, \tag{A.2}$$

$$M_{2n+2} = -m + \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(M_{2n+2} - M_{2q+1})^2}, \quad n = 0, 1, 2, \ldots, \tag{A.3}$$

$$1 = -2 \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(M_{2n+2} - M_{2q+1})^3}, \quad n = 0, 1, 2, \ldots. \tag{A.4}$$

The odd-index Ms are thus determined by equations where the even-index Ms play the role of background data. The roles of odd-index and even-index Ms are interchanged in the complementary sets of equations and m replaced by $-m$.

To analyze the constraints imposed by Eqs. (A.1)-(A.4), we may begin by considering Eqs. (A.1) as representing the intersection condition of two curves. Let us define the two functions

$$f_{+1}(x) \equiv \sum_{q=0}^{\infty} \frac{b_{2q+2}}{(x - M_{2q+2})^2}, \qquad f_{+2}(x) \equiv x - m, \tag{A.5}$$

for x real and where the even-index Ms are assumed to be known. Then, the solutions of Eqs. (A.1) represent the intersection points of the straight line f_{+2} with the curve f_{+1}. We further notice that the function f_{+1} has an infinite number of double poles at the even-index masses with positive residues. Any solution M_{2n+1} should lie between two successive even-index poles. However, in general, the equality $f_{+2} = f_{+1}$ might not lead to such a solution. First, the straight line f_{+2} might not intersect at all the curve f_{+1}, in which case no solution would exist. Or, second, it might cut f_{+1} at two different points between two successive even-index poles. In that case, we would meet instability, since, considering then the complementary sets of equations with the roles of odd and even indices interchanged, and with the number of odd-index solutions doubled, we would generate new solutions of even-index Ms (doubled in number), and so forth. We therefore guess that the only stable and acceptable solutions to Eqs. (A.1) are those corresponding to the situation where the straight line f_{+2} is tangent to the curve f_{+1} between two successive even-index poles. This is precisely the content of Eqs. (A.2). Written as functions of x, they take the form

$$\frac{df_{+2}(x)}{dx} = \frac{df_{+1}(x)}{dx}, \tag{A.6}$$

indicating that the straight line f_{+2} should be tangent to the curve f_{+1} at the positions of the solutions (cf. Fig. 8). Of course, this condition could not be satisfied if the even-index Ms and $\overline{m}$s were arbitrary. The even-index Ms should themselves be solutions of Eqs. (A.3) and (A.4) ensuring the final consistency of all equations.

While the geometrical interpretation of Eqs. (A.1)-(A.4) is very suggestive of the existence of acceptable solutions, one might still question the mutual compatibility of the latter equations. Assuming that the mass parameters $\overline{m}$, or equivalently the residues b, are mainly determined from Eqs. (3.13) or (3.14), one notices that the system of equations (A.1)-(A.4) represent four sets of equations for two sets of unknowns, M_{2n+1} and M_{2n+2}, $n = 0, 1, 2, \ldots$. It is then necessary to check the compatibility of the above equations.

To that end we define two meromorphic functions $h_+(z)$ and $h_-(z)$:

$$h_+(z) \equiv z - m - \sum_{q=0}^{\infty} \frac{b_{2q+2}}{(z - M_{2q+2})^2}, \tag{A.7}$$
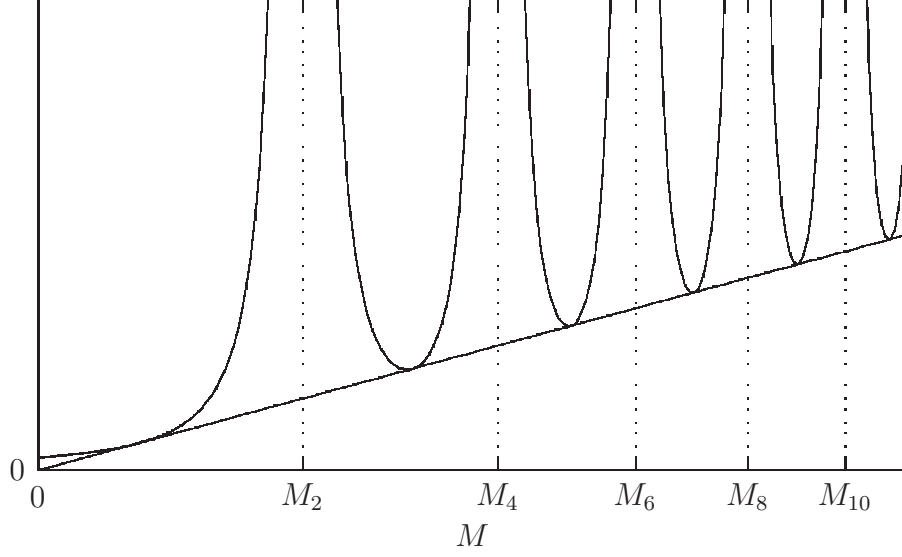
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Figure 8: Graphic representation of Eqs. (A.1) and (A.2) for $m = 0$. The solutions, for odd-index parameters, for given even-index parameter solutions, correspond to the contact points of the straight line with the curves.

$$h_-(z) \; \equiv \; z + m - \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(z - M_{2q+1})^2}. \tag{A.8}$$

The singularities of h_+ are double poles located at the even-index masses, while those of h_- are double poles located at the odd-index masses. The residues b_n are defined in Eqs. (3.4) and are assumed to be finite quantities. Equations (A.1) take the form $h_+(M_{2n+1}) = 0$, $n = 0, 1, 2, \ldots$, which means that h_+ has zeros at the positions of the poles of h_-. Equations (A.2) take the form $\left.\frac{dh_+(z)}{dz}\right|_{z=M_{2n+1}} = 0$, which means that the zeros of h_+ are double. Therefore, the product $h_+ h_-$ is finite at the poles of h_-. A similar argument, using now Eqs. (A.3) and (A.4), also holds at the poles of h_+. In conclusion, the product $h_+(z)h_-(z)$ is an analytic function in the whole complex z-plane. According to Liouville's theorem [29], it is a polynomial of second degree:

$$h_+(z)h_-(z) = z^2 - m^2 + \frac{2\sigma}{\pi}, \tag{A.9}$$

where the right-hand side has been determined from the asymptotic behaviors of h_+ and h_-. It is to be emphasized that the formal calculation of the product of the series (A.7) and (A.8) by exchanging the orders of summation in the double sum would miss the last constant term in the right-hand side of Eq. (A.9); this is due to the weak convergence of the individual series.

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Equation (A.9) indicates that if h_+ is known, then h_- is completely determined from it. It thus provides the proof of compatibility of Eqs. (A.1)-(A.4). For suppose that Eqs. (A.1) and (A.2) are considered on formal grounds as two independent sets of equations determining the two sets of unknowns M_{2n+1} and M_{2n+2}. Then the two remaining equations (A.3) and (A.4) would not bring any new information or constraint since their content is already summarized in Eq. (A.9). On practical grounds, it is still preferable to determine by means of iterative procedures the odd-index masses from Eqs. (A.1) and (A.2) and the even-index masses from Eqs. (A.3) and (A.4).

Relations, involving the residues b_n, are obtained by taking in Eq. (A.9) the limit of z to one of the pole positions of h_+ or h_-. For odd-index poles, for instance, one finds:

$$3b_{2n+1} \sum_{q=0}^{\infty} \frac{b_{2q+2}}{(M_{2n+1} - M_{2q+2})^4} = M_{2n+1}^2 - m^2 + \frac{2\sigma}{\pi}, \qquad (A.10)$$

with a similar relation for the even-index poles. The procedure can be continued with higher-order derivatives at the poles, leading to new relations for the b_ns, but it is not evident whether the derivation operation inside the series remains still valid at higher orders. Equations (A.9) and (A.10) can be used as consistency checks for the solutions that are found.

We next establish Eq. (3.14). The derivation follows similar lines as above. We define two meromorphic functions $f_+(z)$ and $f_-(z)$:

$$f_+(z) \equiv \sum_{q=0}^{\infty} \frac{b_{2q+2}}{(z - M_{2q+2})} + \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(z + M_{2q+1})}, \qquad (A.11)$$

$$f_-(z) \equiv \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(z - M_{2q+1})} + \sum_{q=0}^{\infty} \frac{b_{2q+2}}{(z + M_{2q+2})}. \qquad (A.12)$$

The function f_+ has poles on the positive real axis at even-index masses and on the negative real axis at minus the odd-index masses. f_- has analogous properties with the roles of even-index and odd-index masses interchanged. f_+ and f_- satisfy the following relation:

$$f_-(z) = -f_+(-z). \qquad (A.13)$$

Equations (3.13) can be rewritten in the form

$$f_+(M_{2n+1}) = 0, \qquad f_-(M_{2n+2}) = 0, \qquad n = 0, 1, 2, \ldots . \qquad (A.14)$$

This means that f_+ has zeros at the poles of f_- and f_- has zeros at the poles of f_+. Therefore, the product $f_+(z)f_-(z)$ is free of singularities and is analytic in the whole complex z-plane. According to Liouville's theorem [29], it is a constant, which could be determined from the asymptotic behaviors of f_+ and f_- at infinity:

$$f_+(z) f_-(z) = -\sigma^2. \qquad (A.15)$$

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[Because of the weak convergence of the series defining the functions $f_{\pm}(z)$, it is not possible to obtain the value of the constant by formal calculation of the product $f_+(z)f_-(z)$, the latter operation being invalid under the exchange of the orders of summation in the double series. Asymptotically, the functions $f_{\pm}(z)$ tend to imaginary constants.]

It is then possible to calculate the derivatives of f_+ and f_- at the position of their zeros. Thus:

$$\frac{df_+(z)}{dz}\Big|_{z=M_{2n+1}} = \sigma^2 \lim_{z \to M_{2n+1}} \left(\frac{1}{f_-^2(z)}\frac{df_-(z)}{dz}\right) = -\frac{\sigma^2}{b_{2n+1}}. \tag{A.16}$$

Calculating the expression of the left-hand side and then using Eq. (A.1), we obtain the equation:

$$\frac{\sigma^2}{b_{2n+1}} = M_{2n+1} - m + \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(M_{2n+1} + M_{2q+1})^2}, \tag{A.17}$$

which yields

$$b_{2n+1} = \frac{\sigma^2}{M_{2n+1} - m + \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(M_{2n+1}+M_{2q+1})^2}}. \tag{A.18}$$

Comparison with Eq. (3.4) for odd n then provides the expressions

$$\overline{m}_{2n+1} = \sum_{q=0}^{\infty} \frac{b_{2q+1}}{(M_{2n+1} + M_{2q+1})^2}, \tag{A.19}$$

which correspond to the odd-index parts of Eqs. (3.14). A similar derivation can also be done with the even-index parameters.

Actually, the b_ns could be defined with a different global multiplicative factor than in Eqs. (3.4). The choice done there is the one which satisfies the correct normalization condition of the function F_1, as emphasized in Sec. 3.

Coming back to the functions $h_{\pm}(z)$ [Eqs. (A.7) and (A.8)], it is possible to establish through them a relationship between the functions $iF_1(p^2)$ and $iF_0(p^2)$ [Eqs. (3.1), (3.2) and (3.3)] at $p^2 = 0$. We also include for this analysis the series expansions of their instantaneous limits defined in Eqs. (4.1), (4.2), (4.6), (4.7), (5.24) and (5.25), and which involve the function $\varphi(k)$ parametrizing them (k is the modulus of the space component p_1 of the vector p). Considering the particular values $h_+(0)$ and $h_-(0)$, the derivatives $h'_+(0)$ and $h'_-(0)$ and Eq. (A.9), one finds the following relationships:

$$\varphi'(0)\left(-2m + \frac{2\sigma}{\pi}iF_0(0)\right) = 2 - \frac{4\sigma}{\pi}iF_0(0), \tag{A.20}$$

$$iF_0(0) = \frac{m\pi}{\sigma} + \left[\frac{\pi^2}{4}\varphi'^2(0) + \frac{\pi}{\sigma}(m^2\frac{\pi}{\sigma} - 2)\right]^{1/2}, \tag{A.21}$$

$$iF_1(0) = \frac{\pi}{2\sigma} - \frac{1}{2}\varphi'(0)\left[\frac{\pi^2}{4}\varphi'^2(0) + \frac{\pi}{\sigma}(m^2\frac{\pi}{\sigma} - 2)\right]^{1/2}, \tag{A.22}$$

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where $\varphi'(0) = \left.\frac{\partial\varphi(k)}{\partial k}\right|_{k=0}$. Elimination of $\varphi'(0)$ yields a relationship between $iF_1(0)$ and $iF_0(0)$:

$$iF_1(0) = \frac{\pi}{2\sigma} + \frac{1}{\pi}\left(iF_0(0) - \frac{m\pi}{\sigma}\right)\left[(iF_0(0))^2 - 2iF_0(0)\frac{m\pi}{\sigma} + \frac{2\pi}{\sigma}\right]^{1/2}. \tag{A.23}$$

Numerically, one finds, for $m = 0$, in mass unit of $\sqrt{\sigma/\pi}$, from the series expansion $-\sigma\varphi'(0) = \sum_{n=1}^{\infty} b_n/M_n^2$ [Eq. (5.25)] and the values of M_n and $\overline{m}_n$, $\varphi'(0) = -1.325$, which predicts $iF_1(0) = 1.512$ and $iF_0(0) = 1.528$, in complete agreement with their direct calculation from the series (3.2) and (3.3).

Other relations can also be obtained, like the expressions of the functions φ' and φ in the form of series and alternative expressions of the residues b_n in the form of infinite products of factors. However, these results are not of direct practical relevance for the present calculations and will not be displayed.

A.2 Asymptotic expansions

The asymptotic behaviors of the mass parameters M_n and $\overline{m}_n$ for large n can be studied from Eqs. (5.18) and (5.21). If the quark mass m is itself large, then one has to distinguish two different regions: one for which $\sigma\pi n \gg m^2$ and one for which $\sigma\pi n \ll m^2$. If m is small or null, then the first region may be identified with the region defined by the large n values.

We first consider the first region ($k_i^2 \sim \sigma\pi n \gg m^2$). In the integral (5.18), $\tanh\varphi_i(k_i)$ varies from ± 1 to ∓ 1 when k_i varies between the two mass bounds M_n and M_{n+1}. Therefore, it is the first term of the integrand that dominates and yields a recursion relation that can easily be solved leading to Eq. (3.5). [The other contributions of the integrand vanish at large k_i.] The corresponding behavior of $\overline{m}_n$ [Eq. (3.5)] is obtained from the contribution of the combination $-\frac{\sigma}{2}\varphi'(0) + g_2(M_n)$ to Eq. (5.21), g_2 being defined in Eq. (5.12).

To obtain an estimate of the next-to-leading term in the large n behavior of M_n, it is necessary to specify in more detail the structure of the function $\tanh\varphi_i(k_i)$. Using the analytic continuation of the functions $\sin\varphi(k)$ [Eq. (5.24)] and $\cos\varphi(k)$ [Eq. (5.25)] to the imaginary axis, it is easily seen that $\tanh\varphi_i(k_i)$ can be expressed in terms of the functions f_+ and f_- introduced above [Eqs. (A.11) and (A.12)]:

$$\tanh\varphi_i(k_i) = \frac{f_+(k_i) + f_-(k_i)}{f_+(k_i) - f_-(k_i)}. \tag{A.24}$$

In order to evaluate the integral of $m\tanh\varphi_i(k_i)$ in Eq. (5.18), one can approximate the functions f_+ and f_- by the contributions of the pole terms in M_n and M_{n+1} and take into account the contribution of the remainder, whose role is mainly to ensure the condition

that f_+ (f_-) vanishes at the positions of the poles of f_- (f_+) [Eqs. (A.14)], in the form of effective k_i-dependent residues at the above two poles. Thus, if we are integrating between M_{2n} and M_{2n+1}, we could use the approximations

$$f_+(k_i) \simeq \frac{b_{2n}(k_i - M_{2n+1})}{(M_{2n} - M_{2n+1})(k_i - M_{2n})}, \qquad f_-(k_i) \simeq \frac{b_{2n+1}(k_i - M_{2n})}{(M_{2n+1} - M_{2n})(k_i - M_{2n+1})}, \qquad (A.25)$$

for $M_{2n} \leq k_i \leq M_{2n+1}$, and similar ones for the interval $[M_{2n+1} M_{2n+2}]$.

With the above approximations, the calculation of the integral of $\tanh \varphi_i(k_i)$ can be done explicitly. Since we are considering the region where $m^2 \ll \sigma\pi n$, an expansion in m, up to linear terms in m, can be used. The terms of order zero in m yield alternating signs when passing from one interval to the other and globally do not give significant contributions to the asymptotic domain. The order one terms in m (multiplied with the external m factor) can be treated perturbatively with respect to the leading $\sigma\pi n$ factors. One finds the following behavior:

$$M_n^2 \simeq \sigma\pi(n - n_0) + \left[(2 - \frac{\pi}{2})m^2 - \frac{\sigma}{\pi}\right] \ln(\frac{n}{n_0}) + M_{n_0}^2, \qquad n \geq n_0, \qquad (A.26)$$

where n_0 is the m-dependent lower bound of n to be used. We have also included in the logarithmic term a mass independent contribution coming from the combination $-\frac{\sigma}{2}\varphi'(0) + g_2(k_i)$ in the integrand. The coefficient of the $m^2 \ln(n/n_0)$ term is only approximate.

We next consider the second region, where $\sigma\pi n \ll m^2$. This region is of relevance when heavy quarks or nonrelativistic expansions are considered. The above approximation for $\tanh \varphi_i(k_i)$ can still be used, but now making in the result an expansion in $1/m$. The leading term of the expansion cancels the contribution of the factor k_i of the integrand and the remaining terms are suggestive of a separation of the type $M_n = m + m_n$. The next-to-leading term contributes with alternating signs and is globally negligible. Retaining in the remainder the first leading term and the next one, the final result is:

$$M_n \simeq m \left[1 + 2\left(\frac{\sigma n}{3m^2}\right)^{2/3} + \frac{16}{9\pi}\left(\frac{\sigma n}{3m^2}\right) \ln\left(\frac{3m^2}{\sigma n}\right) \right], \qquad n \leq n_0, \qquad (A.27)$$

where we have chosen for the upper bound of n the same n_0 as in Eq. (A.26), to ensure continuity between the two regions. The coefficients of the last two terms are approximate. The expression of the second term in the right-hand side of the above equation is typical of a nonrelativistic semiclassical energy of a particle of mass m in the presence of a linear confining potential (with a different coefficient), although we do not have in the present problem explicit external forces. The latter result at least justifies the validity of nonrelativistic expansions in domains where the number n respects the upper bound inequality imposed by the value of m. The mass dependence of the nonrelativistic term can

also be obtained by a direct analysis of Eqs. (A.1)-(A.4) with the use of the corresponding expressions of the residues b_n.

The domain corresponding to the interval $[0, M_1]$ does not belong to the asymptotic regions; nevertheless, it is useful to obtain for M_1 an approximate explicit formula. Since that domain is concerned only by one mass, the corresponding approximation to be used for the functions f_+ and f_- will be different. Here, one should keep in f_- its pole term in M_1 and in f_+ its pole term in $-M_1$. No further approximation is needed, since in this case the function $\tanh \varphi_i(k_i)$ correctly vanishes at $k_i = 0$. Explicitly, one has $\tanh \varphi_i(k_i) = -k_i/M_1$ and its integral can be evaluated. Keeping in Eq. (5.16) the factor $(k_i + m \tanh \varphi_i(k_i))$ as the integrand, one obtains for M_1 the formula

$$M_1^2 = \frac{1}{2} \left[m^2 + \sigma\pi + m\sqrt{m^2 + 2\sigma\pi} \right], \tag{A.28}$$

which only provides the order of magnitude of M_1, but could be used in Eq. (A.26) when $n_0 = 1$ (small or null quark mass).

The asymptotic expansions (A.26) and (A.27) were verified in numerical calculations. When $m = 20\sqrt{\sigma/\pi}$, the value $n_0 = 30$ seems to be optimal. For the other masses that are considered, $m = 0., 0.1, 1., 5.$, in unit of $\sqrt{\sigma/\pi}$, n_0 can be set equal to 1 and expansion (A.26) can be used with formuula (A.28) for M_1.

A.3   Numerical resolution

The reconstitution of the functions F_1 and F_0 [Eqs. (3.2) and (3.3)] and of the functions $\sin \varphi$ and $\cos \varphi$ [Eqs. (5.24) and (5.25)] necessitates the knowledge of at least a few hundred mass parameters M_n. Several difficulties, however, are met during the numerical resolution of Eqs. (A.1)-(A.4).

Equations (A.1)-(A.4) represent two sets of compatible equations concerning the same unknown quantities (essentially the M_ns). Their compatibility was established when the whole contributions of the series were taken into account. Their truncation, for computational purposes, introduces a small amount of incompatibility. Furthermore, each of the series has a different rate of convergence under iterative procedures. It is therefore not possible to solve separately and simultaneously these equations without meeting instability problems. The best way is to solve one of the compatible equations, or a combination of the two, and then check the numerical validity of the other.

A second difficulty arises from the fact that the distance between two successive poles in the series decreases with increasing q, becoming of the order of $\sqrt{\sigma\pi/q}$. This means that the corresponding function varies very rapidly between two successive poles and small changes in M_n induce big ones in the equations.

To analyze more quantitatively the accuracy of the calculation, we designate by $e = 0$ a generic equation of the type of Eqs. (A.1) and (A.3), by $e' = 0$ its derivative equation, of the type of Eqs. (A.2) and (A.4), and by x one of the masses M_n. Using Newton's method of iteration [25], equation $e' = 0$ leads to the following iterative relation: $x^{(i+1)} = x^{(i)} - \frac{e'^{(i)}}{e''^{(i)}}$, where e'' is the second-order derivative of the function e with respect to x (M_n) and i is the order of iteration. e'' is negative and for large n its modulus reaches values of the order of $10^4 - 10^6$ (in unit of $\sqrt{\sigma/\pi}$). In these domains, the rate of convergence of the iteration is very slow and violations of the equation $e' = 0$ by factors of unity or 10 actually correspond to very small departures of M_n from its exact value. The analysis of the equation $e = 0$ is more complicated, since here the first derivative of e is itself null or small and one needs to continue the expansion up to the second-order derivative.

A third difficulty arises from the weak convergence of the series of Eqs. (A.1)-(A.4). It requires a detailed treatment of the asymptotic tails.

For the numerical resolution, we have used the combination $e - e'\sqrt{\sigma/\pi}$ of the equations $e = 0$ and $e' = 0$ and checked after each iteration the separate validity of each of them. We have considered 900 active points M_n and similarly for the $\overline{m}_n$s determined simultaneously through Eqs. (3.14). As input values, the asymptotic expansions (A.26) and (A.27) are used. For the asymptotic regions of the various series, 700 background or passive points, from 900 to 1600, are considered using the asymptotic formulas (A.26) and (A.27) with a global scaling factor adjusted after each iteration by continuity at the frontier point $n = 900$. Beyond that region, the contribution of the series is approximated by an integral. High precision on the results is reached after a number of iterations that varies between 10^4 and 5×10^4. The accuracy of the results concerning the first 400 points is better than 10^{-4} and slightly decreases for the remaining points. The uncertainties are mainly due to the contributions of the asymptotic tails. Finally, various consistency relations have been checked: equations (3.13), (A.10), (A.9) (for negative values of z), and the constraint $\sin^2 \varphi(k) + \cos^2 \varphi(k) = 1$.

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